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ERRATUM: "EXISTENCE OF AN UNBOUNDED NODAL HYPERSURFACE FOR SMOOTH GAUSSIAN FIELDS IN DIMENSION $D \geq 3$"

BY HUGO DUMINIL-COPIN[1,a], ALEJANDRO RIVERA[2,b], PIERRE-FRANÇOIS RODRIGUEZ[3,c] AND HUGO VANNEUVILLE[4,d]

[1]*Institut des Hautes Études Scientifiques and Université de Genève,* [a]*duminil@ihes.fr*

[2]*École Polytechnique Fédérale de Lausanne,* [b]*alejandro.rivera@epfl.ch*

[3]*Imperial College London,* [c]*p.rodriguez@imperial.ac.uk*

[4]*CNRS and Université Grenoble Alpes,* [d]*hugo.vanneuville@univ-grenoble-alpes.fr*

This note is an erratum to the paper "Existence of an unbounded nodal hypersurface for smooth Gaussian fields in dimension $d \geq 3$" (*Annals of Probability* **51**(1): 228–276 (January 2023). DOI: 10.1214/22-AOP1594). The published version of this paper contains one error: Proposition 1.12 therein is stated with a sprinkling $R^{-2+\theta_0}$ for some (small) $\theta_0 > 0$, which we cannot afford in Section 5, where this result is applied at mesoscopic scales. We circumvent this issue by proving a stronger version of this proposition, interesting in its own right, which contains no sprinkling. We thus obtain that, for a general class of positively correlated smooth Gaussian fields $f : \mathbb{R}^3 \to \mathbb{R}$ with rapid decay of correlations (including the Bargmann–Fock field), large planar clusters in $\{f \geq 0\} \cap (\mathbb{R}^2 \times \{0\})$ typically belong to clusters in $\{f \geq 0\}$ which are not confined in thin slabs.

The published version of the manuscript [3] contains one error: Proposition 1.12 therein is stated with a sprinkling $R^{-2+\theta_0}$ for some (small) $\theta_0 > 0$, which we cannot afford in [3, Section 5], where this result is applied at mesoscopic scales. The issue can be circumvented by proving the following stronger version of [3, Proposition 1.12], interesting in its own right, which contains no sprinkling. We use the same notations as in [3].

PROPOSITION 1. *Let $d \geq 3$ and let q satisfy Assumption 1.4 for some $\beta > d$. There exist $a, \gamma, c, R_0 > 0$ such that for every $R \geq R_0$, $r \in [r_q, R^\gamma]$ and $\ell \geq 0$,*

$$\mathbb{P}\left[\begin{array}{c}\text{Every continuous path in } \{f_r \geq \ell\} \cap D(2R) \\ \text{from } D(R) \text{ to } \partial D(2R) \text{ belongs to a connected component} \\ \text{of } \{f_r \geq \ell\} \cap (D(2R) \times [0, R^a]) \text{ that intersects } \mathcal{P}_{R^a}\end{array}\right] \geq 1 - \exp(-R^c).$$

As a consequence of this improvement over [3, Proposition 1.12], the arguments in [3, Sections 5,6 and 8] can be somewhat streamlined (although they remain correct as they presently are), by removing this sprinkling. In particular, this entails replacing the level "$2R^{-3/2}$" by "$R^{-3/2}$" in various places, including in the statement of [3, Proposition 5.1], as well as in [3, Definition 7.2, i) and ii)] and [3, Definition 8.2,i)]. Correspondingly, the sprinkling can be removed from the whole discussion in [3, Section 1.3]. In particular, we obtain the following uniqueness property, which improves over [3, (4)] (cf. also [3, Proposition 5.1]):

$$\begin{array}{c}\text{With high probability, for large } R, \text{ any two components of} \\ \text{of } \{f \geq 0\} \cap [0, R]^2 \text{ with diameter } \geq R/100 \\ \text{are connected by a path in } \{f \geq 0\} \cap ([0, R]^2 \times [0, R^a]).\end{array}$$

We now explain how to prove Proposition 1 by modifying the proof of [3, Proposition 1.12]. This proof appears in [3, Section 3] and follows by combining three results, Lemmas

3.2, 3.3 and 3.4. Of these, Lemmas 3.2 and 3.4 remain unchanged. Lemma 3.3 is replaced by the following result. Below the events $E_\ell(R)$ refer to those declared in [3, Definition 3.1] (the events $E_{\ell,\ell'}(R)$ won't be needed anymore).

LEMMA 2. *There exists $\eta > 0$ that depends only on the dimension d such that we have the following as soon as γ, a and $1-b$ are less than η: There exist $c, R_0 > 0$ such that for every $\ell \geq 0$, $R \geq R_0$ and $r \in [r_q, R^\gamma]$, there exists $\ell' \in \mathbb{R}$ such that*

$$\mathbb{P}\left[f_r \in E_\ell(R)\right] \leq \min_{\alpha \in A} \mathbb{P}\left[f_r \in E^\alpha_{\ell'}(R)\right] + R^{-c}.$$

In fact, we prove this result with $\ell' \in [\ell, \ell + CR^{-c}]$ for some $C, c > 0$. We give the proof of Lemma 2 separately below and first conclude the proof of Proposition 1 admitting Lemma 2 and combining with [3, Lemma 3.2+3.4].

PROOF OF PROPOSITION 1. Let us start by observing that Lemma 2 and [3, Lemma 3.4] (applied to the level ℓ' from Lemma 2) and Item 1 of [3, Lemma 3.2] imply that there exist $\gamma, a, \varepsilon, c, R_0 > 0$ such that, for every $R \geq R_0$, $r \in [r_q, R^\gamma]$ and $\ell \geq 0$,

$$\mathbb{P}\left[f_r \in E_\ell(R^{1-\varepsilon})\right] \leq R^{-c}. \tag{1}$$

We now implement a fairly classical renormalization argument. Introduce the event $E_\ell(R, \tau)$ as $E_\ell(R)$ except that the height is τ instead of R^a. Let

$$p(r, \tau, R) = \mathbb{P}[f_r \in E_\ell(R, \tau)].$$

We now claim that there exists $C > 0$ such that for every $L \geq R^\gamma$,

$$p(r, \tau, 7L) \leq (Cp(r, \tau, L))^2.$$

To see this, one can proceed as follows: First, due to independence at distance L, one bounds $p(r, \tau, 7L)$ by the square of the analogous quantity for a $(21L \times 3L)$-rectangle. Now, one covers each $(21L \times 3L)$-rectangle using $40 + 21$ $(3L \times L)$ rectangles and observes that at least one is crossed in the easy direction with a crossing that does not go up to height τ, thus implying the above with $C = 40 + 21$.

By iterating this argument, we obtain that

$$p(r, \tau, 7^k L) \leq C^{2+2^2+\cdots+2^k} p(r, \tau, L)^{2^k} \leq (C^2 p(r, \tau, L))^{2^k}.$$

We now fix $\tau = R^{a(1-\varepsilon)}$ and $L = R^{1-\varepsilon}$. Also, we choose k so that $\frac{1}{7}R^\varepsilon < 7^k \leq R^\varepsilon$ and set $\overline{R} = 7^k R^{1-\varepsilon} \in [R/7, R]$. Using (1), we may find $R_1, c' > 0$ such that for $R \geq R_1$,

$$p(r, R^{a(1-\varepsilon)}, \overline{R}) \leq (C^2 R^{-c})^{2^k} \leq \exp(-R^{c'}).$$

It only remains to observe that for the event in the statement of Proposition 1 to occur (with $2R$ instead of R), there must be one out of $O(1)$ rectangles of size $3\overline{R} \times \overline{R}$ that are crossed by a path which is not in a connected component connected to the top. The claim therefore follows by a union bound. □

It remains to give the proof of Lemma 2. We make frequent reference to [3, Section 3.3].

Proof of Lemma 2. The beginning of the proof is the same as the beginning of [3, Section 3.3] until "...(to prove this, compute the covariance of these Gaussian fields and use the change of variables $u = G_\alpha(y)$)." see the sentence below the display following [3, (7)]. Then the proof continues as follows.

Hence, in order to prove the lemma, it is sufficient to prove that (for γ, a and $1-b$ sufficiently small) there exist $c, R_0 > 0$ such that for any $R \geq R_0$ and $r \in [r_q, R^\gamma]$, there exists $\ell' \in \mathbb{R}$ such that

$$\mathbb{P}[f_r \in E_\ell(R)] \leq \min_{\alpha \in A} \mathbb{P}[h_r^\alpha \in E_{\ell'}(R)] + R^{-c}. \tag{2}$$

Let us now introduce some stability events for percolation clusters, that will enable us to overcome the fact that stability results such as [3, Lemma B.2] cannot be applied to $E_\ell(R)$ (because this event is not a union/intersection of a small number of monotonic events).

If $\ell < \ell'$ are two levels, we let $\mathrm{Stab}_{\ell,\ell'}(R)$ denote the event defined as follows (note that this event only depends on the function restricted to the rectangle $[0,3R] \times [0,R] \subset \mathbb{R}^2 = \mathbb{R}^2 \times \{0\}^{d-2}$): a function $u \in C(\mathbb{R}^d)$ belongs to $\mathrm{Stab}_{\ell,\ell'}(R)$ if every connected component of $\{u \geq \ell\} \cap ([0,3R] \times [0,R])$ that contains a continuous path from $[0,3R] \times \{0\}$ to $[0,3R] \times \{R\}$ also contains such a path γ with the further property that $u_{|\gamma} \geq \ell'$.

Let us now make the following observation: For every $\delta > 0$ and every $u, v \in C(\mathbb{R}^d)$, at least one of the following properties does not hold:

- $u \in E_\ell(R)$;
- $u \in \mathrm{Stab}_{\ell,\ell+2\delta}(R)$;
- $\|u - v\|_{\infty,[0,3R]\times Q\times\{0\}^{d-3}} \leq \delta$;
- $v \notin E_{\ell+\delta}(R)$.

As a result, for every $\delta > 0$ we have

$$\begin{aligned}\mathbb{P}[f_r \in E_\ell(R)] \leq{}& \min_{\alpha \in A} \mathbb{P}[h_r^\alpha \in E_{\ell+\delta}(R)] \\ &+ \max_{\alpha\in A} \mathbb{P}\big[\|f_r - h_r^\alpha\|_{\infty,[0,3R]\times Q\times\{0\}^{d-3}} \geq \delta\big] + \mathbb{P}\big[f_r \notin \mathrm{Stab}_{\ell,\ell+2\delta}(R)\big].\end{aligned}$$

We thus obtain that Lemma 2 is a consequence of the following two lemmas:

LEMMA 3. *Fix some $\varepsilon \in (0,1)$. There exists a constant $\theta > 0$ that depends only on ε such that the following holds as soon as $\gamma < 1 - \varepsilon$: There exists $R_0 > 0$ such that for every $\ell \geq 0$, $R \geq R_0$ and $r \in [r_q, R^\gamma]$,*

$$\mathbb{P}\big[f_r \notin \mathrm{Stab}_{\ell,\ell+\delta}(R)\big] \leq R^{-\theta},$$

where $\delta := R^{-2+\theta}$.

LEMMA 4. *For every $\theta > 0$, there exists $\eta > 0$ (that depends only on θ and on the dimension d) such that the following holds as soon as γ, a and $1-b$ are less than η: There exist $c, R_0 > 0$ such that for every $R \geq R_0$ and $r \in [r_q, R^\gamma]$,*

$$\max_{\alpha \in A} \mathbb{P}\big[\|f_r - h_r^\alpha\|_{\infty,[0,3R]\times Q\times\{0\}^{d-3}} \geq R^{-2+\theta}\big] \leq \exp(-R^c).$$

Although not stated as such, Lemma 4 is proved in [3, Section 3.3]: see [3, (8)] (in particular, the proof uses [3, Claim 3.7], which remains valid).

PROOF OF LEMMA 3. In this proof, we use the notion of stratified gradient of some function $u \in C^2(\mathbb{R}^2)$ with respect to the rectangle $[0,3R]\times[0,R]$. The stratified gradient $\nabla^s_x u$ is defined as the usual (2-dimensional) gradient if x does not belong to the boundary of the rectangle; it is defined as the one-dimensional gradient $\nabla_x(u_{|L})$ if x belongs to some side L of the rectangle excluding corners, and $\nabla^s_x u := 0$ if x is a corner of the rectangle.

Let $\delta > 0$.

CLAIM 1. *Assume that $f_r \notin \mathrm{Stab}_{\ell,\ell+\delta}(R)$. Then, there exist a connected component C of $\{f_r \geq \ell\} \cap ([0,3R]\times[0,R])$ and a point $x \in C$ such that:*

- *C contains a continuous path from $B_R := [0,3R]\times\{0\}$ to $T_R := [0,3R]\times\{R\}$;*
- *$f_r(x) \in [\ell, \ell+\delta]$ and $\nabla^s_x f_r = 0$.*

PROOF. Let K denote the union of all connected components of $\{f_r \geq \ell\} \cap ([0,3R]\times[0,R])$ that contain a continuous path from T_R to B_R. Our aim is to prove the following claim: Assume that there is no $x \in K$ such that $\nabla^s_x f_r = 0$ and $f_r(x) \in [\ell, \ell+\delta]$. Then, every connected component of K contains a continuous path γ from T_R to B_R such that $f_r|\gamma \geq \ell + \delta$.

Let us prove this claim. To this purpose, let K^ε (resp. $\overline{K}^\varepsilon$) denote the open (resp. closed) ε-neighborhood of K. We fix some $\varepsilon > 0$ such that $(f_r)_{|K^{2\varepsilon}\setminus K} < \ell$ and, by using smooth Urysohn's lemma (applied to the compact set $(K^{2\varepsilon})^c$ included in the open set $(\overline{K}^\varepsilon)^c$, both seen as subsets of $[0,3R]\times[0,R]$), we construct a function $\widetilde{f}_r \in C^2(\mathbb{R}^2)$ such that

- $(\widetilde{f}_r)_{|K^\varepsilon} = (f_r)_{|K^\varepsilon}$ and
- $(\widetilde{f}_r)_{|K^c} < \ell$.

We note that that there is no $x \in [0,3R]\times[0,R]$ such that $\widetilde{f}_r(x) \in [\ell, \ell+\delta]$ and $\nabla^s_x \widetilde{f}_r = 0$, and we apply a result from stratified Morse theory to $\widetilde{f}_r$ as follows: By [4, Proposition in Section 3.2 of Part I], there exists a homeomorphism φ from $K = \{\widetilde{f}_r \geq \ell\} \cap ([0,3R]\times[0,R])$ to $L := \{\widetilde{f}_r \geq \ell+\delta\} \cap ([0,3R]\times[0,R])$ such that both φ and φ^{-1} send a point of B_R (resp. T_R) on a point of B_R (resp. T_R). The existence of a homeomorphism between K and L implies that the number of connected components of L is the same as the number of connected components of K. Since every connected component of L is included in a connected component of K, we obtain that every connected component of K contains a component of L. Moreover, the property of φ implies that every component of L contains a path from B_R to T_R. This concludes the proof of the desired result for $\widetilde{f}_r$, which implies the desired result for f_r. □

Let us now tile the rectangle $[0,3R]\times[0,R]$ with $\asymp R^2$ unit squares S_i and let us note that, for every $h > 0$, there exists $C_h > 0$ that depends only on h and q such that

$$\forall i, \quad \mathbb{P}[\exists x \in S_i, \nabla^s_x f_r = 0 \text{ and } f_r(x) \in [\ell, \ell+\delta]] \leq C_h \delta^{1-h}. \tag{3}$$

This is for instance written at the end of the proof of [5, Lemma 7] (applied to $m = 2$, $\beta = 1$, $\tau = \delta$ and $t = h/3$ – let us note that the fact that the constant C_h above – of which the reader can find an expression in [5] – is uniform in r is a consequence of classical Gaussian estimates such as Dudley's theorem and the BTIS inequality, both applied to the Gaussian field $(f_r(x), \nabla^s_x f_r, (\nabla^s)^2_x f_r)_{x\in S_i}$, see [1, 2]).

Let us end the proof by using (3) as well as the RSW theorem – [3, Theorem 2.4] (the use of the RSW theorem here is the only reason why Lemma 3 and Proposition 1 are stated for levels $\ell \geq 0$). By the RSW theorem and the independence between sets at distance greater than r, there exists a constant $c > 0$ that depends only on ε such that, if $\gamma < 1 - \varepsilon$ and $r \in [r_q, R^\gamma]$, we have

$$\forall i, \quad \mathbb{P}\big[\exists \text{ a cont. path in } \{f_r \geq \ell\} \text{ from } S_i^r \text{ to } T_R \text{ and such a path from } S_i^r \text{ to } B_R\big] \leq R^{-c}, \tag{4}$$

where S_i^r is the set of all points in $\mathbb{R}^2$ at distance less than r from S_i.

Let $\delta = R^{-2+\theta}$ as in the statement of the lemma. By using (3), (4) and the independence between sets at distance greater than r, we have

$$\forall i, \quad \mathbb{P}\big[\exists x \in S_i \text{ as in Claim 1}\big] \leq C_h \delta^{1-h} R^{-c} = C_h R^{(-2+\theta)(1-h)-c}.$$

Choosing for instance $h = \theta = c/1000$ and summing over i imply the desired result. □

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REFERENCES

[1] ADLER, R. J. and TAYLOR, J. E. (2007). *Random Fields and Geometry*. Springer.
[2] AZAÏS, J.-M. and WSCHEBOR, M. (2009). *Level sets and extrema of random processes and fields*. John Wiley & Sons.
[3] DUMINIL-COPIN, H., RIVERA, A., RODRIGUEZ, P.-F. and VANNEUVILLE, H. (2023). Existence of an unbounded nodal hypersurface for smooth Gaussian fields in dimension $d \geq 3$. *The Annals of Probability* **51** 228 – 276. https://doi.org/10.1214/22-AOP1594
[4] GORESKY, M. and MACPHERSON, R. (1988). *Stratified Morse theory*. Springer.
[5] NAZAROV, F. and SODIN, M. (2016). Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. *Journal of Mathematical Physics, Analysis, Geometry* **12** 205–278.