Noise sensitivity without spectrum: a simple example

Hugo Vanneuville
based on a joint work with Vincent Tassion

Abstract

The sharp noise sensitivity theorem for planar percolation has been proven in [GPS10] by C. Garban, G. Pete and O. Schramm via a geometric study of the Fourier spectrum of percolation events. In [TV20], V. Tassion and I have given a non-spectral proof of this result, inspired by H. Kesten's proof of the so-called scaling relations [Kes87]. The core of this approach is the study of a PDE satisfied by pivotal probabilities. In this note, I present this proof in the simple case of the iterated majority function and then explain briefly why this strategy also applies to percolation. The goal is only pedagogical in the sense that one could show sharp noise sensitivity of the iterated majority function via a more direct non-spectral approach.

Contents

1 Sharp noise sensitivity for the iterated majority

Let $n \geq 1$. We consider the hypercube $\{0,1\}^n$ equipped with the uniform probability measure

$$\mathbf{P} = \left(\frac{\delta_0 + \delta_1}{2} \right)^{\otimes n}.$$

Let $t \in [0,1]$, let $X \sim \mathbf{P}$ and let $Y \sim \mathbf{P}$ be obtained from X by resampling independently each coordinate with probability t. For every $f : \{0,1\}^n \to \mathbb{R}$ we let

$$\mathrm{COV}_t(f) = \mathrm{Cov}(f(X), f(Y)).$$

The notion of noise sensitivity has been introduced by I. Benjamini, G. Kalai and O. Schramm in [BKS99]. It is defined as follows - for more about this notion, we refer to the two books [GS14, O'D14].

Definition 1.1 ([BKS99])**.** Let m_n be a sequence of positive integers. We say that a sequence $g_n : \{0,1\}^{m_n} \to \{0,1\}$ is noise sensitive if there exists a sequence $t_n \to 0$ such that

$$\mathrm{COV}_{t_n}(g_n) \to 0.$$

A function from $\{0,1\}^n$ to $\{0,1\}$ is called a Boolean function. In this note, we study noise sensitivity properties of the 3-iterated majority function. Let us define this function as well as the majority function.

- The majority function $\mathrm{Maj}_n : \{0,1\}^n \to \{0,1\}$ is defined for every odd $n \geq 1$ by
$$\mathrm{Maj}_n = 1_{\{\text{there are more 1's than 0's}\}}.$$
- The 3-iterated majority function $\mathrm{IMaj}_n : \{0,1\}^n \to \{0,1\}$ is defined for every n of the form 3^k (with $k \geq 1$) iteratively as follows:
  i) $\mathrm{IMaj}_3 = \mathrm{Maj}_3$,
  ii) $\mathrm{IMAJ}_{3^k}(x) = \mathrm{IMaj}_{3^{k-1}}(\mathrm{Maj}_3(x_1,x_2,x_3),\ldots,\mathrm{Maj}_3(x_{3^k-2},x_{3^k-1},x_{3^k}))$.

Each time we work with the iterated majority, the parameter n will implicitly be of the form 3^k for some $k \geq 1$.

In all the note, we will see $\mathrm{IMAJ}_n(x)$ as the value assigned to the root of some 3-ary labelled tree: Consider a rooted tree of depth k such that every vertex has three children (except the leaves). We see the leaves as the integers $1,\ldots,n = 3^k$ and we assign the value x_i to the leaf i. Moreover, we assign a value to each other vertex of the tree iteratively as follows: the value assigned to a vertex is obtained by applying IMAJ_3 to its three children. Then, $\mathrm{IMAJ}_n(x)$ is the value assigned to the root, see Figure 1.

Notation 1.2. We let ρ denote the root of the tree and if v, w are two vertices of the tree we let $[v,w)$ denote the injective path from v to w with the convention that $v \in [v,w)$ and $w \notin [v,w)$.

One can show that IMaj_n is noise sensitive while Maj_n is not. In order to explain the goal of this note, we need the following definition.

Definition 1.3. We say that i is pivotal for $f : \{0,1\}^n \to \{0,1\}$ and $x \in \{0,1\}^n$ if changing x_i modifies $f(x)$.

We let

$$f_n = 1_{\{1 \text{ is piv. for } \mathrm{IMAJ}_n\}} \qquad \text{and} \qquad \alpha_n = \mathbf{E}[f_n].$$

We observe that $f_n = 1$ if and only if, for each vertex $v \in [1,\rho)$, the two siblings of v have different values (which is for instance the case on Figure 1). By independence,

$$\alpha_n = 2^{-k} = n^{-\log(2)/\log(3)} = n^{-0.63\ldots}. \tag{1}$$

The reader can keep in mind that

$$n\alpha_n = \mathbf{E}[\text{number of pivotal coordinates for } \mathrm{IMAJ}_n] = n^{0.37\ldots}$$

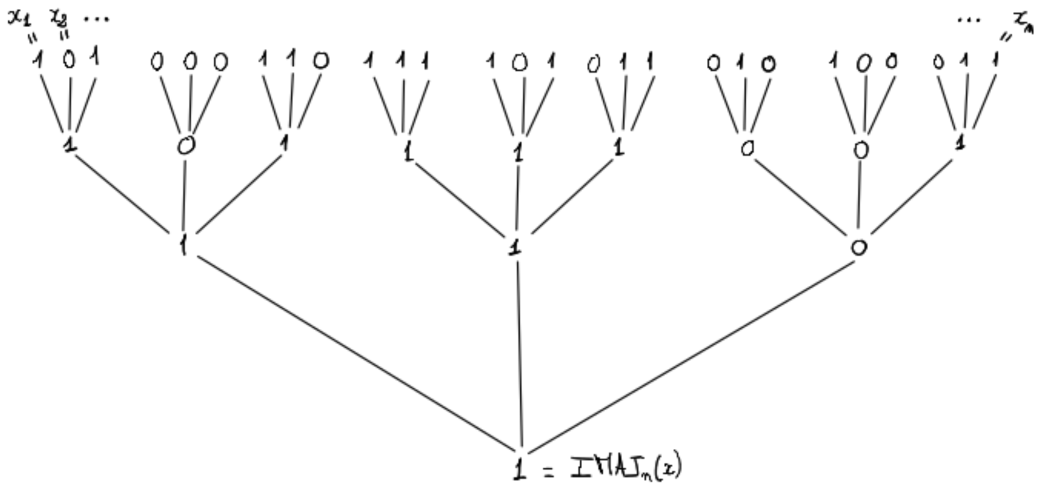

Figure 1: $\mathrm{IMAJ}_n(x)$ is the value assigned to the root of a tree of depth k.

goes to $+\infty$ polynomially fast, but much less faster than in the case of the majority function:

$$n\mathbf{P}[1 \text{ is piv. for } \mathrm{MAJ}_n] = \frac{n}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}} = \sqrt{n}(\sqrt{2/\pi} + o(1)).$$

Let us also note that $tn\alpha_n$ is the expected number of coordinates that are pivotal in X and that are resampled when one defines Y. It might be natural to expect that $\mathrm{COV}_t(\mathrm{IMAJ}_n)$ is small if and only if $tn\alpha_n$ is large. This is not true for general Boolean functions since this is not true for MAJ_n, but this is true for IMAJ_n and we have the following result.

Proposition 1.4 (Sharp noise sensitivity). *If $t_n n\alpha_n \to +\infty$ then*

$$\mathrm{COV}_{t_n}(\mathrm{IMAJ}_n) \to 0.$$

If $t_n n\alpha_n \to 0$ then

$$\mathrm{COV}_{t_n}(\mathrm{IMAJ}_n) \to 1/4.$$

Such a "sharp noise sensitivity result" has been proven in the case of percolation events by C. Garban, G. Pete and O. Schramm in [GPS10]. Previously, I. Benjamini, G. Kalai and O. Schramm [BKS99] have proven that planar percolation is noise sensitive and Schramm and Steif [SS10] have proven that percolation is sensitive to some polynomially small noise. All these proofs rely on the study of the spectral decomposition of percolation events. In [TV20], V. Tassion and I have given a non-spectral proof of sharp noise sensitivity of percolation, inspired by H. Kesten's proof of the so-called scaling relations [Kes87]. In the present note, we present this proof in the simple case of the iterated majority. The goal is only pedagogical in the sense that one could prove Proposition 1.4 via a more direct non-spectral approach. At the end of the note, we state the sharp noise sensitivity result for planar percolation and explain briefly why the strategy of the present note is well adapted to its proof.

The main object of the proof is

$$\pi_n(t) = \mathbb{E}[f_n(X) f_n(Y)],$$

which is the probability that 1 is pivotal both in X and in Y.

2 Differential formulas

The proof is based on differential formulas satisfied by $\mathrm{COV}_t(\mathrm{IMAJ}_n)$ and $\pi_n(t)$. In order to state and prove such formulas, we consider the following discrete gradient operator

$$\nabla_i f = \frac{f \circ \sigma_i^1 - f \circ \sigma_i^0}{2},$$

where for every $i \in \{1, \ldots, n\}$ and $\epsilon \in \{0,1\}$, $\sigma_i^\epsilon : \{0,1\}^n \to \{0,1\}^n$ is the operator that assigns ϵ to the i^{th} coordinate and leaves the others unchanged. Recall that $X \sim \mathbf{P}$ and that $Y \sim \mathbf{P}$ is obtained from X by resampling independently each coordinate with probability t.

Lemma 2.1. *Let $n \geq 1$ and $f : \{0,1\}^n \to \mathbb{R}$. We have*

$$-\frac{d}{dt}\mathrm{COV}_t(f) = -\frac{d}{dt}\mathbb{E}[f(X)f(Y)] = \sum_{i=1}^n \mathbb{E}[\nabla_i f(X) \nabla_i f(Y)] \geq 0.$$

Before proving the lemma, let us state a corollary and deduce the second part of Proposition 1.4. Recall that

$$f_n = 1_{\{1 \text{ is piv. for } \mathrm{IMAJ}_n\}} \qquad \text{and} \qquad \pi_n(t) = \mathbb{E}[f_n(X) f_n(Y)].$$

Corollary 2.2. *The functions $t \mapsto \mathrm{COV}_t(\mathrm{IMAJ}_n)$ and $t \mapsto \pi_n(t)$ are non-increasing. More precisely, we have*

$$-\frac{d}{dt}\mathrm{COV}_t(\mathrm{IMAJ}_n) = \frac{1}{4} n \pi_n(t) \tag{2}$$

and

$$0 \leq -\pi_n'(t) \leq -\pi_n'(0) = \frac{1}{4}\sum_{i=1}^n \mathbf{P}[i \text{ is piv. for } f_n]. \tag{3}$$

Proof. (2) comes from Lemma 2.1, the fact that $\mathbb{E}[\nabla_i \mathrm{IMAJ}_n(X) \nabla_i \mathrm{IMAJ}_n(Y)]$ does not depend on i, and the following observation: since $\mathrm{IMAJ}_n(x)$ is non-decreasing in x, we have

$$\nabla_1(\mathrm{IMAJ}_n) = \frac{f_n}{2}.$$

In order to prove (3), we apply Lemma 2.1 to f_n and obtain the following:

$$\begin{aligned} 0 \leq -\pi_n'(t) \leq \sum_{i=1}^n \mathbb{E}\big[|\nabla_i f_n|(X) |\nabla_i f_n|(Y)\big] &\leq \frac{1}{2}\sum_{i=1}^n \mathbb{E}\big[|\nabla_i f_n|(X)\big] \\ = \frac{1}{4}\sum_{i=1}^n \mathbf{P}[i \text{ is piv. for } f_n] &= \sum_{i=1}^n \mathbb{E}\big[|\nabla_i f_n|^2(X)\big] = -\pi_n'(0). \end{aligned}$$

□

Proof of the second part of Proposition 1.4. The proof of the second part is very general. By Corollary 2.2,

$$0 \leq \mathrm{COV}_{t_n}(\mathrm{IMAJ}_n) - \frac{1}{4} = \mathrm{COV}_{t_n}(\mathrm{IMAJ}_n) - \mathbf{Var}(\mathrm{IMAJ}_n)$$
$$= \frac{1}{4}\int_0^{t_n} n\pi_n(s)ds \leq \frac{t_n n \pi_n(0)}{4} = \frac{t_n n \alpha_n}{4}.$$

□

Proof of Lemma 2.1. Let us first note that since the law of Y does not depend on t we have

$$\frac{d}{dt}\mathrm{COV}_t(f) = \frac{d}{dt}\mathbb{E}[f(X)f(Y)].$$

Let us now show the rest of the lemma. To this purpose, let $X \sim \mathbf{P}$, let $Z \sim \mathbf{P}$ independent of X, let $(U_1, \dots, U_n)$ be uniform in $[0,1]^n$ independent of (X,Z), and for each $\mathbf{t} \in [0,1]^n$ define $Y(\mathbf{t}) \sim \mathbf{P}$ as follows:

$$Y_i(\mathbf{t}) = \begin{cases} X_i \text{ if } U_i \leq \mathbf{t}_i \\ Z_i \text{ if } U_i > \mathbf{t}_i \end{cases}.$$

In particular, $(X, Y(t,\dots,t))$ has the same law as (X,Y) from the statement. The lemma is then a direct consequence of the chain rule and of the following claim:

$$-\frac{\partial}{\partial \mathbf{t}_i}\mathbb{E}[f(X)f(Y(\mathbf{t}))] = \mathbb{E}[\nabla_i f(X)\nabla_i f(Y(\mathbf{t}))] \geq 0.$$

Let us show this claim. To this purpose, we observe that for each i,

$$f(x) = (2x_i - 1)\nabla_i f(x) + \mathbf{E}_i f(x),$$

where

$$\mathbf{E}_i f = \frac{f \circ \sigma_i^1 + f \circ \sigma_i^0}{2}.$$

Given some $\delta \in [-1,1]$ such that $\mathbf{t}_i + \delta \in [0,1]$, we let $\mathbf{t}'_j = \mathbf{t}_j$ if $j \neq i$ and $\mathbf{t}'_i = \mathbf{t}_i + \delta$. Moreover, we write $Y = Y(\mathbf{t})$ and $Y' = Y(\mathbf{t}')$. By using that both $\nabla_i f(x)$ and $\mathbf{E}_i f(x)$ do not depend on x_i (and that $\mathbb{E}[Y'_i - Y_i] = 0$) we obtain that

$$\begin{aligned}\mathbb{E}\big[f(X)f(Y')\big] - \mathbb{E}\big[f(X)f(Y)\big] &= 2\mathbb{E}\big[f(X)(Y'_i - Y_i)\nabla_i f(Y)\big] \\ &= 2\mathbb{E}\big[\big((2X_i - 1)\nabla_i f(X) + \mathbf{E}_i f(X)\big)(Y'_i - Y_i)\nabla_i f(Y)\big] \\ &= 4\mathbb{E}\big[X_i(Y'_i - Y_i)\big]\mathbb{E}\big[\nabla_i f(X)\nabla_i f(Y)\big] \\ &= -\delta\mathbb{E}\big[\nabla_i f(X)\nabla_i f(Y)\big].\end{aligned}$$

As a result,

$$-\frac{\partial}{\partial \mathbf{t}_i}\mathbb{E}[f(X)f(Y(\mathbf{t}))] = \mathbb{E}[\nabla_i f(X)\nabla_i f(Y(\mathbf{t}))].$$

It only remains to show that this quantity is non-negative. To this purpose, we introduce the following coupling of the law of $(X, Y(\mathbf{t}))$: Let $\mathbf{s} \in [0,1]^n$ defined by $1 - \mathbf{t}_i = (1 - \mathbf{s}_i)^2$. Moreover, let $\widetilde{X} \sim \mathbf{P}$, let $\widetilde{W} \sim \mathbf{P}$ be obtained from X

by resampling independently every coordinate i with probability $\mathbf{s}_i$ and let $\widetilde{Y}$ be obtained from $\widetilde{W}$ by resampling independently every coordinate i by probability $\underset{\sim}{\mathbf{s}_i}$. Then, $(\widetilde{X}, \widetilde{Y})$ is a coupling of the law of $(X, Y(\mathbf{t}))$. Moreover, if we condition on $\widetilde{W}$, then $\widetilde{X}$ and $\widetilde{Y}$ are independent and have the same law. As a result,

$$\mathbb{E}[\nabla_i f(X)\nabla_i f(Y(\mathbf{t}))] = \mathbb{E}[\nabla_i f(\widetilde{X})\nabla_i f(\widetilde{Y})] = \mathbb{E}\big[\mathbb{E}[\nabla_i f(\widetilde{X}) \mid \widetilde{W}]^2\big] \geq 0. \qquad \square$$

Remark 2.3. Lemma 2.1 is also the consequence of general results for Markov semigroups, see for instance [BGL13]. As noticed in [GP20] where P. Galicza and G. Pete study sparse reconstruction of Boolean functions, Lemma 2.1 is an analogue for Boolean functions of the covariance formulas which are at the core of S. Chatterjee's theory of superconcentration and chaos [Cha14]. Let us point out that R. Eldan and R. Gross [EG20] have also used differential formulas in order to prove general noise sensitivity results, with a martingale point of view.

3 Proof of sharp noise sensitivity

The proof of Proposition 1.4 relies on three properties satisfied by $\pi_n(t)$. Let us state and prove these properties. Before this, let us just make one comment about our conventions about constants: below, c, C are positive and finite constants that are not necessarily the same at each occurence and depend neither on the time parameters (i.e. t, s, ...) nor on the space parameters (i.e. n, m, ...). Recall that when we work with the iterated majority (so in all the present section), n and m are implicitly of the form 3^k for some $k \geq 1$.

3.1 Three properties satisfied by $\pi_n(t)$

The three properties are the following:

$$-\pi_n'(0) \leq Cn\alpha_n^2, \tag{P1}$$

$$\int_0^1 n\pi_n(t)dt \leq C, \tag{P2}$$

$$\text{if } n \geq m \text{ and } s \geq t \text{ then } \frac{\pi_n(s)}{\pi_n(t)} \leq C\frac{\pi_m(s)}{\pi_m(t)}. \tag{P3}$$

Before proving them, let us discuss quickly these properties. By Corollary 2.2,

$$0 \leq -\pi_n'(0) = \frac{1}{4}\sum_{i=1}^n \mathbf{P}[i \text{ is piv. for } f_n]. \tag{4}$$

As a result, (P1) implies that if we condition on $\{f_n = 1\}$ then the expected number of pivotal coordinates for f_n is at most of the same order as the expected number of pivotal coordinates for IMAJ_n.

Let us now discuss (P2). Recall that $n\pi_n(t)$ is the expected number of coordinates that are pivotal in both X and Y. Since π_n is non-increasing, (P2) implies for instance that

$$n\pi_n(t) \leq \frac{1}{t}\int_0^t n\pi_n(s)ds \leq \frac{1}{t}\int_0^1 n\pi_n(s)ds \leq \frac{C}{t}. \tag{5}$$

Thus, even if the expected number of pivotal coordinates (which equals $n\alpha_n = n\pi_n(0)$) goes to $+\infty$, the expected number of coordinates that are pivotal in both X and Y is bounded uniformly in n for every $t > 0$.

Let us finally discuss (P3). This "quasi-monotonicity property" implies that (up to a constant) the larger the scale is, the faster π_n decays. This will enable us to deduce some properties about π_n from properties satisfied at lower scales.

Proof of (P1). The proof relies on (4). Recall that $f_n = 1$ if and only if, for each $w \in [1, \rho)$, the two siblings of w have different values (recall that ρ is the root). Let $i \in \{2, \dots, n\}$, let v be youngest common ancestral vertex of 1 and i and let u be the child of v who is also an ancestral vertex of i. Then, the event $\{i \text{ is piv. for } f_n\}$ is the event that

- for all $w \in [1, \rho) \setminus \{v\}$, the two siblings of w have different indices,
- changing the color of i modifies the color of u, i.e. for all $w \in [i, u)$, the two siblings of w have different indices.

As a result, if $\ell \geq 1$ is the smallest integer such that $i \leq 3^\ell$, we have

$$\mathbf{P}[i \text{ is piv. for } f_n] = \frac{\alpha_n}{2} \frac{\alpha_{3^\ell}}{2}.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{P}[i \text{ is piv. for } f_n] &= \sum_{\ell=1}^{k} (3^\ell - 3^{\ell-1}) \frac{\alpha_n}{2} \frac{\alpha_{3^\ell}}{2} \\ &\leq \frac{n\alpha_n^2}{4} \sum_{\ell=1}^{k} \frac{3^\ell}{n} \frac{\alpha_{3^\ell}}{\alpha_n} \overset{(1)}{=} \frac{n\alpha_n^2}{4} \sum_{\ell=1}^{k} (3/2)^{\ell-k} \leq C n \alpha_n^2. \end{aligned}$$

□

Proof of (P2). By Corollary 2.2, we even have $\int_0^1 \frac{n\pi_n(t)}{4} dt = \mathbf{Var}(\mathrm{IMAJ}_n) = \frac{1}{4}$ (but we have stated the property in a more general form). □

Proof of (P3). Let $m \leq n$ and let v be the youngest common ancestral vertex of $1, \dots, m$. We couple IMAJ_n and IMAJ_m by letting $\mathrm{IMAJ}_m(x_1, \dots, x_m)$ be the value assigned to v. We observe that there exists a Boolean function $f_{m,n} : \{0,1\}^{n-m} \to \{0,1\}$ such that

$$f_n(x_1, \dots, x_n) = f_m(x_1, \dots, x_m) f_{m,n}(x_{m+1}, \dots, x_n).$$

We let

$$\pi_{m,n}(t) = \mathbb{E}[f_{m,n}(X_{m+1}, \dots, X_n) f_{m,n}(Y_{m+1}, \dots, Y_n)]$$

where as before $X \sim \mathbf{P}$ and Y is obtained from X by resampling independently every coordinate with probability t. By independence,

$$\frac{\pi_n(t)}{\pi_m(t)} = \pi_{m,n}(t) \overset{\text{Lem. 2.1}}{\geq} \pi_{m,n}(s) = \frac{\pi_n(s)}{\pi_m(s)}.$$

As a result, (P3) even holds with $C = 1$. □

3.2 Superlinearity (and analytic proof of sharp noise sensitivity)

Using (P1), (P2) and (P3) and Corollary 2.2, we prove Proposition 1.4. Now, the proof is only analytic. Let

$$\varepsilon_n = \frac{1}{2C_0 n\alpha_n}$$

where $C_0 > 0$ is some fixed constant such that (P1) holds with $C = C_0$.

Proposition 3.1 (Superlinear decay of π_n on the right of ε_n)**.** *If $\varepsilon_n \leq t \leq s \leq 1$ then*

$$\frac{\pi_n(s)}{\pi_n(t)} \leq C\Big(\frac{t}{s}\Big)^{1+c}.$$

Before proving Proposition 3.1, let us deduce Proposition 1.4 from it. Recall that we have already shown the second part of Proposition 1.4.

Proof of the first part of Proposition 1.4. If $t_n \geq \varepsilon_n$ then

$$\begin{aligned}\mathrm{COV}_{t_n}(\mathrm{IMAJ}_n) = \int_{t_n}^1 n\pi_n(s)ds &\overset{\text{Prop. 3.1}}{\leq} C\int_{t_n}^1 n\pi_n(\varepsilon_n)\left(\frac{\varepsilon_n}{s}\right)^{1+c} ds \\ &\leq Cn\alpha_n\varepsilon_n\left(\frac{\varepsilon_n}{t_n}\right)^c \leq C\left(\frac{\varepsilon_n}{t_n}\right)^c,\end{aligned}$$

where in the second to last inequality we have used that π_n is non-increasing and $\pi_n(0) = \alpha_n$. This ends the proof since $(\varepsilon_n/t_n)^c$ goes to 0 if $t_n n\alpha_n \to +\infty$. □

Let us now prove Proposition 3.1. Let us start by using (P1) to prove the following lemma.

Lemma 3.2 (Stability on the left of ε_n)**.** $\pi_n(\varepsilon_n) \geq \alpha_n/2$.

Proof. We have

$$\pi_n(\varepsilon_n) = \alpha_n + \int_0^{\varepsilon_n} \pi_n'(s)ds \overset{\text{Cor. 2.2}}{\geq} \alpha_n + \varepsilon_n\pi_n'(0) \overset{\text{(P1)}}{\geq} \alpha_n - C_0\varepsilon_n n\alpha_n^2 = \alpha_n/2. \quad \square$$

By Lemma 3.2, $\pi_n(t)$ is of the same order as $1/\varepsilon_n$ on $[0,\varepsilon_n]$. But by (P2) $\int_0^1 n\pi_n(t)dt$ is uniformly bounded, so $\pi_n(t)$ must decay fast just to the right of ε_n. For instance we have the following:

$$\frac{\pi_n(t)}{\pi_n(\varepsilon_n)} \overset{(5)}{\leq} \frac{C}{nt\pi_n(\varepsilon_n)} \overset{\text{Lem. 3.2}}{\leq} C\frac{\varepsilon_n}{t}.$$

We are little more quantitative in the following lemma, and improve this linear decay by a $\sqrt{\log}$ factor.

Lemma 3.3. *Let $\lambda \geq 1$. If $\lambda\varepsilon_n \leq 1$ then*

$$\frac{\pi_n(\lambda\varepsilon_n)}{\pi_n(\varepsilon_n)} \leq \frac{C}{\lambda\sqrt{\log\lambda}}.$$

Proof. We first note that (since π_n is non-increasing) it is sufficient to show that, for λ larger than some constant, there exists $\mu \in \left[\frac{\lambda}{\sqrt{\log \lambda}}, \lambda\right]$ such that

$$\frac{\pi_n(\mu\varepsilon_n)}{\pi_n(\varepsilon_n)} \leq \frac{1}{\mu \log \mu}.$$

If this were not the case, then we would have

$$\begin{aligned} C \overset{(P2)}{\geq} \int_{\frac{\lambda}{\sqrt{\log \lambda}}\varepsilon_n}^{\lambda\varepsilon_n} n\pi_n(t)dt &> \int_{\frac{\lambda}{\sqrt{\log \lambda}}\varepsilon_n}^{\lambda\varepsilon_n} n\pi_n(\varepsilon_n)\frac{1}{\frac{t}{\varepsilon_n}\log(\frac{t}{\varepsilon_n})}dt \\ &= n\pi_n(\varepsilon_n)\varepsilon_n \int_{\frac{\lambda}{\sqrt{\log \lambda}}}^{\lambda} \frac{1}{t\log t}dt \overset{\text{Lem. 3.2}}{\geq} c\log\log\sqrt{\log \lambda}, \end{aligned}$$

which cannot be true if λ is sufficiently large. □

We have obtained that π_n decays much more than linearly just on the right of ε_n. In order to obtain that this is true everywhere on the right of ε_n, we use (P3), which enables us to compare the decay of π_n (close to some t, say) with the decay of π_m for some $m \leq n$. The suitable choice for m will satisfy $t \simeq \varepsilon_m$.

Proof of Proposition 3.1. Let $\varepsilon_n \leq t \leq s \leq 1$. We first note that it is sufficient to prove the result for s of the form $\lambda^k t$ where k is some integer and $\lambda > 0$ is fixed but chosen at the end of the proof. We let

$$\ell_j = \min\{n : \varepsilon_n \leq \lambda^j t\}$$

and we note that

$$\ell_j \leq n$$

and

$$\varepsilon_{\ell_j} \leq \lambda^j t \leq C\varepsilon_{\ell_j} \text{ (because } \varepsilon_n \leq C\varepsilon_{3n}). \tag{6}$$

We write

$$\frac{\pi_n(s)}{\pi_n(t)} = \frac{\pi_n(\lambda^k t)}{\pi_n(t)} = \prod_{j=0}^{k-1} \frac{\pi_n(\lambda^{j+1}t)}{\pi_n(\lambda^j t)}$$

and estimate each term of the product separately:

$$\frac{\pi_n(\lambda^{j+1}t)}{\pi_n(\lambda^j t)} \overset{(P3)}{\leq} \frac{\pi_{\ell_j}(\lambda^{j+1}t)}{\pi_{\ell_j}(\lambda^j t)} \overset{(6)}{\leq} \frac{\pi_{\ell_j}(\frac{\lambda}{C}\varepsilon_{\ell_j})}{\pi_{\ell_j}(\varepsilon_{\ell_j})} \overset{\text{Lem. 3.3}}{\leq} \frac{C}{\lambda\sqrt{\log\frac{\lambda}{C}}}.$$

Let us fix $\lambda > 0$ sufficiently large so that $\frac{C}{\lambda\sqrt{\log\frac{\lambda}{C}}} = \lambda^{1+c(\lambda)}$ for some $c(\lambda) > 0$. Altogether,

$$\frac{\pi_n(s)}{\pi_n(t)} \leq \left(\frac{C}{\lambda\sqrt{\log\frac{\lambda}{C}}}\right)^k = \lambda^{-k(1+c(\lambda))} = \left(\frac{t}{s}\right)^{1+c(\lambda)}. \qquad \square$$

Remark 3.4. Superlinear decay in time of π_n on the right of ε_n implies superlinear decay in space: If $m \le n$ and $t \ge \varepsilon_m$ then

$$\frac{\pi_n(t)}{\pi_m(t)} \overset{(P2)}{\le} \frac{\pi_n(\varepsilon_m)}{\pi_m(\varepsilon_m)} \overset{\text{Lem. 3.2}}{\le} 2\frac{\pi_n(\varepsilon_m)}{\alpha_m} \overset{\pi_n\searrow}{\le} 2\frac{\alpha_n}{\alpha_m}\frac{\pi_n(\varepsilon_m)}{\pi_n(\varepsilon_n)}$$
$$\overset{\text{Prop. 3.1}}{\le} C\frac{\alpha_n}{\alpha_m}\Big(\frac{\varepsilon_n}{\varepsilon_m}\Big)^{1+c} = C\frac{m}{n}\Big(\frac{m\alpha_m}{n\alpha_n}\Big)^c \le C\Big(\frac{m}{n}\Big)^{1+c}.$$

4 A few words about percolation (and about estimating more precisely $\pi_n(t)$)

4.1 Sharp noise sensitivity for planar percolation

A. Why does this strategy apply to percolation? Let us first state the sharp noise sensitivity theorem for planar percolation. Consider the (regular planar) triangular grid, let $\mathcal{V}$ denote its set of vertices and, given some $x \in \{0,1\}^{\mathcal{V}}$, color each site v black if $x_v = 1$ and white if $x_v = 0$. Let us use notations similar to those used in the other sections: For any $\mathcal{V}' \subset \mathcal{V}$, we equip the hypercube $\{0,1\}^{\mathcal{V}'}$ with the uniform measure that we denote by $\mathbf{P}$. Moreover, given some $t \in [0,1]$, we let $X \sim \mathbf{P}$ and we let $Y \sim \mathbf{P}$ be obtained from X by resampling the color of each vertex independently with probability t.

We define two Boolean functions:

- Let R_n be a rhombus drawn on the triangular grid containing n^2 vertices with two distinguished opposite sides that we call left and right sides. Also, let $\mathcal{V}_n = \mathcal{V} \cap R_n$ and let $\mathrm{Cross}_n : \{0,1\}^{\mathcal{V}_n} \to \{0,1\}$ denote the indicator function of the black left-right crossing event of R_n. Note that by symmetry we have

$$\mathbf{E}[\mathrm{Cross}_n] = 1/2.$$

- Let $\mathcal{V}'_n = \mathcal{V} \cap [-n,n]^2$ and let $f_n : \{0,1\}^{\mathcal{V}'_n} \to \{0,1\}$ denote the indicator funcion of the event that there are four paths of alternating color from the origin to $\partial[-n,n]^2$. Moreover, let

$$\alpha_n = \mathbf{E}[f_n] \qquad \text{and} \qquad \pi_n(t) = \mathbb{E}[f_n(X)f_n(Y)].$$

By using the so-called separation of arms theory of H. Kesten [Kes87] (see below), the theory of SLE process of O. Schramm [Sch00] and conformal invariance of percolation proven by S. Smirnov [Smi01], S. Smirnov and W. Werner [SW01] have proven that

$$\alpha_n = n^{-3/4+o(1)}.$$

Without the use of SLE techniques and conformal invariance, it is known how to prove that (see for instance Chapter 6 of [GS14] and references therein)

$$cn^{-(2-c)} \le \alpha_n \le Cn^{-(1+c)}. \tag{7}$$

In particular, $n^2\alpha_n$ goes to $+\infty$ polynomially fast.

The sharp noise sensitivity theorem for planar percolation is the following result.

Theorem 4.1 ([GPS10]). *If* $t_n n^2 \alpha_n \to +\infty$ *then*

$$\mathrm{COV}_{t_n}(\mathrm{Cross}_n) \to 0.$$

If $t_n n^2 \alpha_n \to 0$ *then*

$$\mathrm{COV}_{t_n}(\mathrm{Cross}_n) \to 1/4.$$

B. Why is this result analogous to Proposition 1.4? Let $v \in \mathcal{V}_n$. The event that v is pivotal for Cross_n is the event that there are two black paths and two white paths from v to the left and right sides and the top and bottom sides respectively. Contrary to IMAJ_n, the probability of this event depends on v. However, by the so-called separation of arms properties by Kesten [Kes87], one can prove that, if $v \in \mathcal{V}_n$ is at distance at least $n/4$ (for instance) of ∂R_n,

$$c\alpha_n \leq \mathbf{P}[v \text{ is piv. for } \mathrm{Cross}_n] \leq C\alpha_n. \tag{8}$$

Let us say a few more words about this separation of arms theory. This theory essentially implies that if we know that there are some paths of prescribed color from some mesoscopic or microscopic scale m to some macroscopic scale n, then with non-negligible probability, we can ask that the paths are well-separated and reach some prescribed macroscopic zones. For instance, if we know that there exist four paths of alternating colors from some site v in the bulk of R_n to scale n, then the probability that these paths are connected to some prescribed sides of R_n is non-negligible. This implies (8). We refer to [Kes87, Wer07, Nol08, SS10] for more about these techniques.

Morever, one can prove that for all $v \in \mathcal{V}_n$,

$$\mathbf{P}[v \text{ is piv. for } \mathrm{Cross}_n] \leq C\alpha_n.$$

In particular,

$$cn^2\alpha_n \leq \mathbf{E}[\text{number of pivotal vertices for } \mathrm{Cross}_n] \leq Cn^2\alpha_n,$$

so Theorem 4.1 is indeed the analogue of Proposition 1.4.

C. Why does the strategy from the other sections enable to prove Theorem 4.1? Let us ignore boundary effects - which are essentially technical problems - and let us assume (even if this is not true when v is close to ∂R_n) that for all $v \in \mathcal{V}_n$

$$c\pi_n(t) \leq \mathbf{P}[v \text{ is piv. for } \mathrm{Cross}_n \text{ both in } X \text{ and } Y] \leq C\pi_n(t).$$

Once again (for v not close to ∂R_n), this is a consequence of the separation of arms theory initiated by H. Kesten. More precisely, this is a consequence of its extension to coupled percolation configurations in [GPS10]. Using this, we obtain that (P1), (P2) and (P3) still hold in the context of percolation (with n^2 instead of n in (P2)):

- The proof of (P2) is the same.
- The proof of (P3) goes as follows: For $1 \leq m \leq n$, $f_{m,n}$ is the indicator of the event that there are four paths of alternating colors from $\partial[-m,m]^2$ to $\partial[-n,n]^2$. By using the separation of arms theory, one can prove the following quasi-multiplicativity property:

  $$c\pi_n(t) \leq \pi_m(t)\pi_{m,n}(t) \leq C\pi_n(t), \quad \text{where} \quad \pi_{m,n}(t) = \mathbb{E}[f_{m,n}(X)f_{m,n}(Y)].$$

  Using this, the proof of (P3) is the same as in the case of the iterated majority.

- Concerning the proof of (P1), we refer to Section 4.2.

With (P1), (P2) and (P3) in hand, the proof of Theorem 4.1 is now essentially the same as the analytic proof from Section 3.2.

Remark 4.2. The proof of noise sensitivity of percolaion in earlier works deeply relies on upper bounds for pivotal events. In particular, I. Benjamini, G. Kalai and O. Schramm [BKS99] show that a sequence of Boolean functions $g_n : \{0,1\}^{m_n} \to \{0,1\}$ is noise sensitive if $\sum_{i=1}^n \mathbf{P}[i \text{ is piv. for } g_n]^2$ goes to 0. As a result, noise sensivity for Cross_n is a consequence of the upper bound $\alpha_n \leq Cn^{-(1+c)}$ from (7). On the contrary, one of the main inputs of the strategy presented in this note is the lower bound from (7), which implies that $n^2\alpha_n$ goes to $+\infty$ (and more generally the multiscale lower bound (11)). Upper bounds such as $\alpha_n \leq Cn^{-(1+c)}$ or the more general multiscale property

$$\text{if } m \leq n, \text{ then } \frac{\alpha_n}{\alpha_m} \leq C\Big(\frac{m}{n}\Big)^{1+c} \tag{9}$$

are rather consequences of some of our intermediate results (more precisely, of the analogues of Remark 3.4 and (P2) for percolation). (9) has first been proven by C. Garban in the appendix of [SS11].

4.2 Kesten's PDE

A. Let us start this section with a few words about the proof of (P1) for planar percolation. As in the case of the iterated majority (i.e. Corollary 2.2) we have the following in the case of percolation:

$$-\pi_n'(0) = \frac{1}{4} \sum_{v \in \mathcal{V}_n'} \mathbf{P}[v \text{ is piv. for } f_n].$$

Moreover, by the separation of arms theory, and once again if we ignore boundary effects, one can show that if v is at distance of order m from the origin, then

$$c\alpha_n\alpha_m \leq \mathbf{P}[v \text{ is piv. for } f_n] \leq C\alpha_n\alpha_m.$$

As a result,

$$c\alpha_n \sum_{m=1}^n m\alpha_m \leq -\pi_n'(0) \leq C\alpha_n \sum_{m=1}^n m\alpha_m. \tag{10}$$

Now, the proof of (P3) is the same as in the case of the iterated majority since it is known (see for instance Chapter 6 of [GS14] and references therein) that if $m \leq n$ then

$$\frac{\alpha_n}{\alpha_m} \leq C\Big(\frac{m}{n}\Big)^{2-c}. \tag{11}$$

B. The inequalities (10) is Kesten's PDE at time 0. Let us say a few words about this. In the study of noise sensitivity, the separation of arms theory actually implies that (10) holds at any $t \in [0,1]$:

$$c\pi_n(t) \sum_{m=1}^n m\pi_m(t) \leq -\pi_n'(t) \leq C\pi_n(t) \sum_{m=1}^n m\pi_m(t). \tag{12}$$

We see (12) as a PDE for $\pi_n(t)$ on the space/time domain $[0,1] \times \mathbb{N}^*$. The proof of the so-called "scaling relations" by H. Kesten [Kes87] relies on an analogous PDE in the "near-critical phase". By plugging Proposition 3.1 in (12), one can compute precisely the order of $\pi_n(t)$ and prove the following theorem (which also holds for the iterated majority if we replace $n^2\alpha_n$ by $n\alpha_n$ in the definition of $\ell(t)$).

Theorem 4.3 ([TV20]). *Let* $\ell(t) = \min\{n : \frac{1}{t} \leq n^2\alpha_n\}$.

- *If* $n \geq \ell(t)$, *then* $c\pi_n(t) \leq \alpha_{\ell(t)} \left(\frac{\alpha_n}{\alpha_{\ell(t)}}\right)^2 \leq C\pi_n(t)$,
- *If* $n \leq \ell(t)$, *then* $c\pi_n(t) \leq \alpha_n \leq C\pi_n(t)$.

We refer to [TV20] for the (short, once we have Proposition 3.1 in hand) proof of Theorem 4.3. Once the order of $\pi_n(t)$ is known, one can compute precisely the order of many percolation quantitaties (as functions of quantities "at time 0" such as α_n).

References

[BGL13] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and geometry of Markov diffusion operators.* Springer Science & Business Media, 2013.

[BKS99] Itai Benjamini, Gil Kalai, and Oded Schramm. Noise sensitivity of Boolean functions and applications to percolation. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 1999.

[Cha14] Sourav Chatterjee. *Superconcentration and related topics.* Springer, 2014.

[EG20] Ronen Eldan and Renan Gross. Concentration on the Boolean hypercube via pathwise stochastic analysis. *arXiv preprint 1909.12067*, 2020.

[GP20] Pál Galicza and Gábor Pete. Sparse reconstruction in spin systems I: iid spins. *arXiv preprint 2010.10483*, 2020.

[GPS10] Christophe Garban, Gábor Pete, and Oded Schramm. The Fourier spectrum of critical percolation. *Acta Mathematica*, 2010.

[GS14] Christophe Garban and Jeffrey E. Steif. *Noise sensitivity of Boolean functions and percolation.* Cambridge University Press, 2014.

[Kes87] Harry Kesten. Scaling relations for 2D-percolation. *Communications in Mathematical Physics*, 1987.

[Nol08] Pierre Nolin. Near-critical percolation in two dimensions. *Electronic Journal of Probability*, 2008.

[O'D14] Ryan O'Donnell. *Analysis of boolean functions.* Cambridge University Press, 2014.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 2000.

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *Comptes Rendus de l'Académie des Sciences*, 2001.

[SS10] Oded Schramm and Jeffrey E. Steif. Quantitative noise sensitivity and exceptional times for percolation. *Annals of Mathematics*, 2010.

[SS11] Oded Schramm and Stanislav Smirnov. On the scaling limits of planar percolation. with an appendix by Christophe Garban. *Annals of Probability*, 2011.

[SW01] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Mathematical Research Letters*, 2001.

[TV20] Vincent Tassion and Hugo Vanneuville. Noise sensitivity of percolation via differential inequalities. *arXiv preprint 2011.04572*, 2020.

[Wer07] Wendelin Werner. Lectures on two-dimensional critical percolation. *IAS Park City Graduate Summer School*, 2007.